ETH zürich

Institute for Dynamic Systems and Control

Control Systems 1

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17.10.2025

Welcome!

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Today

- Repetition Session …
- Theory Recap
  - Transfer Function and Laplace Transform
- Q&A Session / Done

Repetition Session

How do we divide the time output (response) to get an explicit expression for it

A) For one response we set x(0) = 0

C) Boundary Condition + Forced

B) Initial Condition + Forced

D) For one response we set u(t) = 0

How do we divide the time output (response) to get an explicit expression for it

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Time Response

Remember, we want to find an **explicit expression** for our time response.
We divide the response into initial condition response (**IC**) and forced response (**F**).
This is possible, because we are looking at linear systems (**superposition**).

$$\boxed{y = y_{\mathrm{IC}} + y_{\mathrm{F}}},$$

$$\left.\begin{aligned} x_{\mathrm{IC}}(0) &= x_0, \\ u_{\mathrm{IC}}(t) &= 0, \quad t \geq t_0, \end{aligned}\right. \quad \to \quad y_{\mathrm{IC}}; \qquad + \qquad \left.\begin{aligned} x_{\mathrm{F}}(0) &= 0, \\ u_{\mathrm{F}}(t) &= u(t), \quad t \geq t_0, \end{aligned}\right. \quad \to \quad y_{\mathrm{F}}.$$

Time Response Solution n - Order System

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\,B\,u(\tau)\,\mathrm{d}\tau,$$

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{IC}} + \underbrace{C\int_0^t e^{A(t-\tau)}\,B\,u(\tau)\,\mathrm{d}\tau + D\,u(t)}_{\text{F}}.$$

With A, B, C and D being the matrices from the LTI SISO State-Space

$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned}}$$

True or False?

There are, nevertheless, two important cases where the matrix exponential admits a closed-form expression: When the matrix is diagonalizable, and when it is in Neumann form.

True

False

True or False?

There are, nevertheless, two important cases where the matrix exponential admits a closed-form expression: When the matrix is diagonalizable, and when it is in ~~Neumann~~ (Jordan) form.

True

False

Matrix Exponential

How do we compute the matrix exponentials of the form e^{At}?

Either the matrix is already in one of the convenient forms:

- **Diagonal Matrix**

$$\boxed{\exp\left(\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} t\right) = \begin{bmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}}$$

- **Jordan Normal Form**

$$\boxed{\exp\left(\begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix} t\right) = e^{\lambda t}\begin{bmatrix}1 & t \\ 0 & 1\end{bmatrix}}$$

In the other case we have to get it into our desired representation (if **diagonalizable**):

$$\tilde{A} = T^{-1}AT = diag(\lambda_1, ..., \lambda_n)$$

Matrix diagonalisieren (Basiswechsel in Eigenbasis)

1. Bestimme die Eigenwerte λ_i und die Eigenvektoren v_i
2. Die Matrix $D = diag(\lambda_1, \dots, \lambda_n)$ ist eine Diagonalmatrix mit den Eigenwerten auf der Diagonalen.
3. Die Matrix $T = (v_1, \dots, v_n)$ hat die Eigenvektoren als Spalten **(Gleiche Reihenfolge wie bei D!)**.
4. Bestimme T^{-1}. Falls EV orthonormal $T^{-1} = T^T$.

What is important for the stability assessment of a system?

A) Imaginary part of eigenvalues of A

C) Diagonal Matrix A

B) Real part of eigenvalues of A

D) Eigenvalues of A, B, C and D

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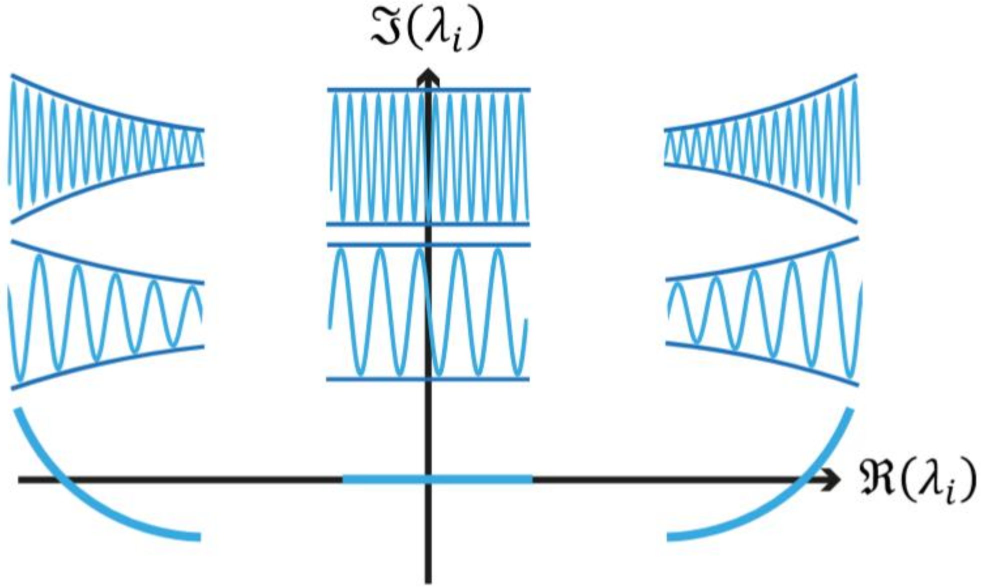
Stability Classification



Asymptotically Stable: State converges to zero for bounded initial conditions and zero input.

$$\mathrm{Re}(s) < 0 \text{ for all } \lambda_i.$$

Lyapunov Stable: State will remain bounded for bounded initial conditions and zero input.

$$\mathrm{Re}(s) \leq 0 \text{ for all } \lambda_i.$$

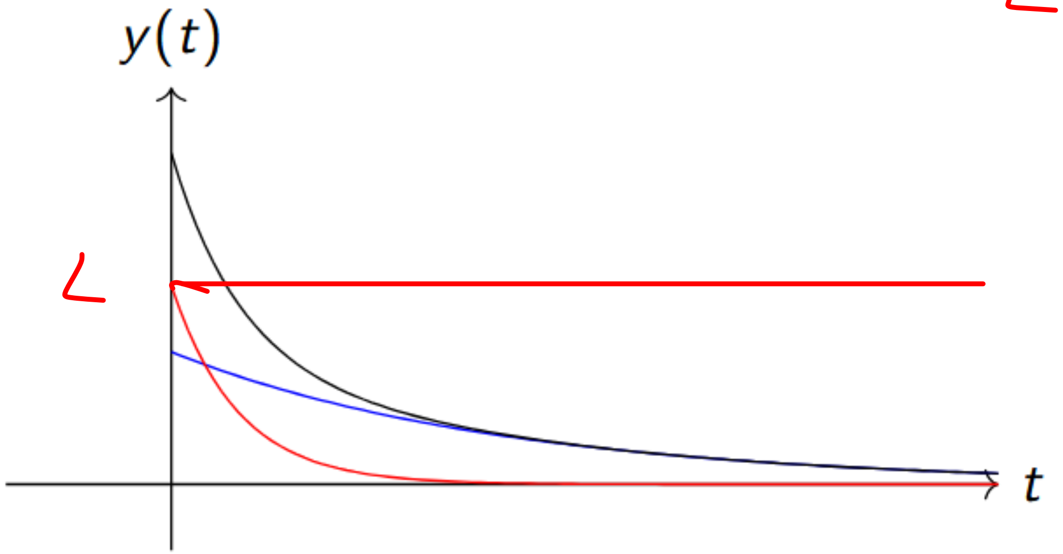
BIBO Stable: Output remains bounded for every bounded input.

In Linear Systems: **Asymptotically stable → Lyapunov stable**
Asymptotically stable → BIBO stable

Example

$\lambda_1 = 0$

$$y(t) = Ce^{At}x_0 = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = \underbrace{c_1 e^{\lambda_1 t} x_{0,1}}_{= L} + c_2 e^{\lambda_2 t} x_{0,2}$$



Theory Recap

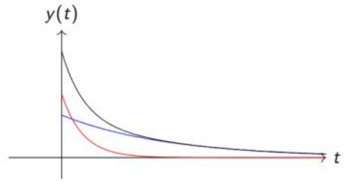
Transfer Function and Laplace Transform

Transfer Function

$$\dot{x} = f(x(t), u(t))$$
$$y = h(x(t), u(t))$$

Example

$$y(t) = Ce^{At}x_0 = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = c_1 e^{\lambda_1 t} x_{0,1} + c_2 e^{\lambda_2 t} x_{0,2}$$



Last week we talked about how the initial condition response behaves. But what about the forced response and the convolution integral?

$$u(t) \Rightarrow u(\tau)$$

$$C \int_0^t e^{A(t-\tau)} Bu(\tau) \mathrm{d}\tau + Du(t).$$

This is much harder to interpret. For intuition, we start with elementary and simple inputs, and since our system is linear, we can later advantage from a superposition.

$$u(t) = e^{st}, \quad u(\tau) = e^{s\tau}$$

Let's just say we want to analyze the systems response to an exponential e^{st}. Why?

$$C \int_0^t e^{A(t-\tau)} \cdot Bu(\tau)\, d\tau + Du(t)$$

Laplace Transform

Time Domain $\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$ Frequency / s Domain

Remember for the a fourier analysis, we say that every periodic function can be decomposed as a sum of sinusoids.

For **exponential functions** however, we need to take on this problem by applying the Laplace Transform. It not only represents the function as sum of sinusoids, but **also** as a sum of exponentials!

We can, especially, **write almost any function as a sum of complex exponentials**, which leads us to the realization that we can create any input function as a sum of the more simple exponentials.

For a better understanding about Laplace and Frequency Domain, you might want to take a look at additional sources, such as videos on youtube. As recommended on my website, **Brian Douglas** has some nice videos, but also **3Blue1Brown** or **Zach Star.**

Laplace Transform

The formulas for the Laplace transform are given on the right.
However I would recommend using the tables
below for most calculations. You know it from Ana III…

$$F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t,$$

$$f(t) = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}\mathrm{d}s.$$

$f(t)$	$\mathscr{L}f(t) = F(s)$	$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$	$\sin kt$	$\frac{k}{s^2+k^2}$
$e^{at}f(t)$	$F(s-a)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$		
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$
$\delta(t)$	1		
$\delta(t-a)$	e^{-sa}	$\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
$\frac{d^n}{dt^n}\delta(t)$	s^n	e^{at}	$\frac{1}{s-a}$

Transfer Function again

Lets now see what happens when we plug in the exponential input e^{st} into our time response:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t). \quad \rightarrow \quad y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Be^{s\tau}\mathrm{d}\tau + De^{st},$$

…..

Can be seen in Lecture

$$\boxed{y(t) = \underbrace{Ce^{At}\left[x_0 - (sI - A)^{-1}B\right]}_{\text{Transient response}} + \underbrace{\left[C(sI - A)^{-1}B + D\right]e^{st}}_{\text{Steady state response}}.}$$

Transfer function

$$y(t) = \underbrace{Ce^{At}\left[x_0 - (sI-A)^{-1}B\right]}_{\text{Transient response}} + \underbrace{\boxed{\left[C(sI-A)^{-1}B+D\right]}\, e^{st}}_{\text{Steady state response}}.$$

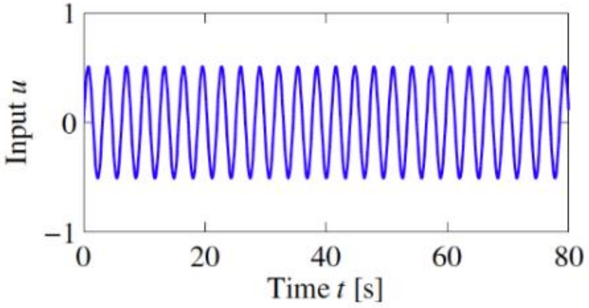
G(s)

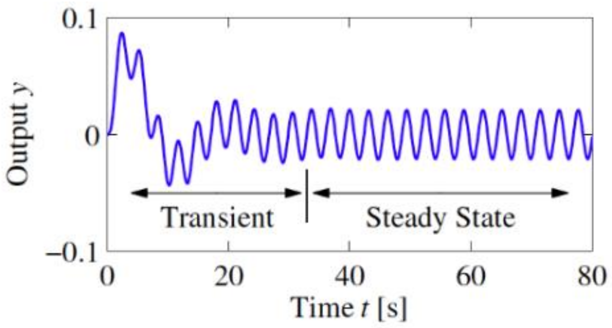
Let us define the **transfer function** G(s) as the following: $G(s) = C(sI-A)^{-1}B + D.$

The steady state response is therefore defined as: $y_{ss}(t) = G(s)e^{st}$

ss = steady state

When a system is asymptotically stable, the transient response (with matrix A in exponential) goes to zero! See below:





Transfer Function

When deriving the transfer with the laplace transform with find:

$$
\begin{aligned}
Y(s) &= C\left[(sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)\right] + DU(s) \\
&= C(sI - A)^{-1}x_0 + \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)}U(s).
\end{aligned}
$$

It can be seen that is is nearly the same as before, but now all is in Laplace Domain.

Now taking the inverse of our equation, we and achieve our normal solution:

$$
y(t) = \mathcal{L}^{-1}\{G(s)U(s)\}.
$$

Annotations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{Ax(t)\} + \mathcal{L}\{Bu(t)\}$$

$$sX(s) - x_0 = AX(s) + BU(s)$$

$$(s\mathbb{I} - A)X(s) = x_0 + BU(s)$$

$$X(s) = (s\mathbb{I} - A)^{-1}x_0 + (s\mathbb{I} - A)^{-1}BU(s)$$

Example

$$G(s) = C(sI - A)^{-1}B + D$$

Consider the pendulum system,d escribes by the matrices below, derive the transfer function:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0.$$

$$\boxed{\begin{bmatrix} 1 & 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}}_{} \Bigg)^{-1} \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} + D}$$

$$\begin{bmatrix} s & -1 \\ 2 & s+1 \end{bmatrix}^{-1}$$

$$\|$$

$$\frac{1}{s^2+s+2} \begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2+s+2} \begin{bmatrix} 0.2 \\ 0.2s \end{bmatrix}$$

$$\Rightarrow G(s) = \frac{0.2}{s^2+s+2}$$

Transfer Function

In general, the transfer function G(s) (Laplace domain) can be written as:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D$$

Poles are the roots (Nullstellen) of the denominator

Zeros are the roots of the numerator

There is another form called the **Controllable Canonical Form**. It provides a minimal dimension model to fully describe our system.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \dots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}, \quad D = [d];$$

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$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D$$

Q17 **(1 Points)**

Given: $n = 5$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -4 & -1 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{array}{ccccc} [\frac{1}{2} & -2 & 0 & -\frac{1}{3} & 0] \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{array}, D = 0.$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

$-\frac{1}{3} \neq$

- ☐ ~~$G(s) = \frac{\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3}}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$~~
- ☐ $G(s) = \frac{-\frac{1}{3}\cdot s^3 - 2\cdot s + \frac{1}{2}}{s^5 + 6\cdot s^4 + s^3 + 4\cdot s^2 + 2}$
- ☐ ~~$G(s) = \frac{-\frac{1}{3}\cdot s^3 - 2\cdot s + \frac{1}{2}}{s\cdot(s^4 + 6\cdot s^3 + s^2 + 4\cdot s + 2)}$~~ $= \underbrace{-a_0 \cdot s}_{\text{wrong}}$
- ☐ ~~$G(s) = \frac{s\cdot(\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3})}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$~~

$-\frac{1}{3} \neq$

A)

C)

B)

D)

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Q17 (1 Points)

Given:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -4 & -1 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{2} & -2 & 0 & -\frac{1}{3} & 0 \end{bmatrix}, D = 0.$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

- ☐ $G(s) = \frac{\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3}}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$
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- ☐ $G(s) = \frac{s\cdot(\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3})}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$

A)

C)

B)

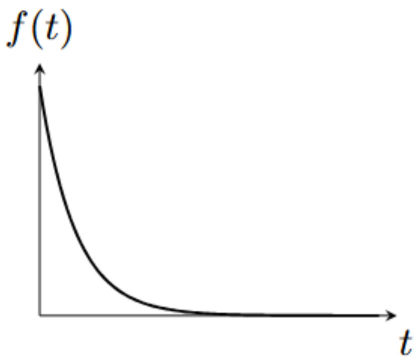
D)

Laplace Transform

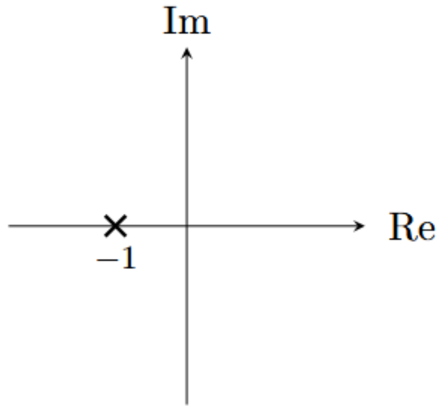
Consider the function $f(t) = e^{-t}$, a simple time decaying exponential.
By applying the laplace transform we find it's laplace domain function F(s). Look at the transform table!

$$\boxed{f(t) = e^{-t}} \qquad \boxed{F(s) = \frac{1}{s+1}}$$

$$\frac{1}{s+1}$$



$$\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$$



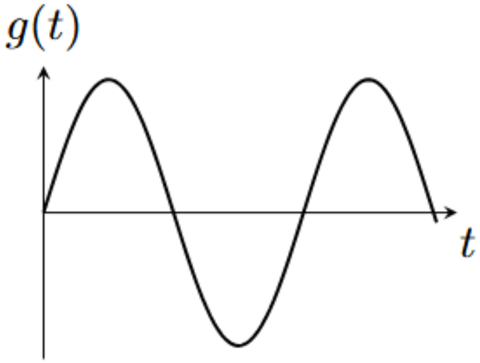
For clarity, we will use **crosses** to indicate peaks in F(s) instead of plotting the entire function.
Peaks arise wherever the denominator goes to zero and the magnitude to infinity. $s \to -1$

Laplace Transform

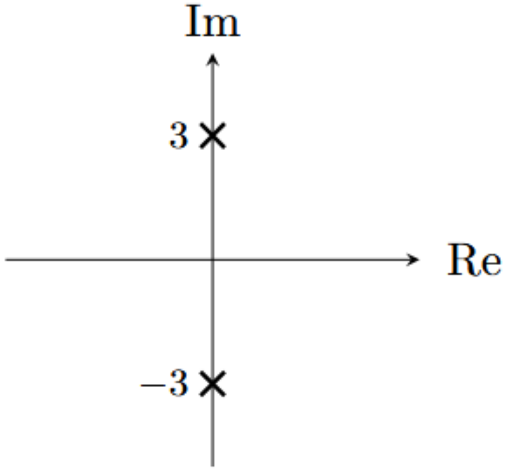
Let’s look at another example. What happens when the function is made up of exponentials with imaginary parts?

$$\boxed{g(t) = \sin(3t)}$$

$$\boxed{G(s) = \frac{3}{s^2+9} = \frac{3}{(s+3j)(s-3j)},}$$



$$\xrightarrow{\mathcal{L}}$$
$$\xleftarrow[\mathcal{L}^{-1}]{}$$

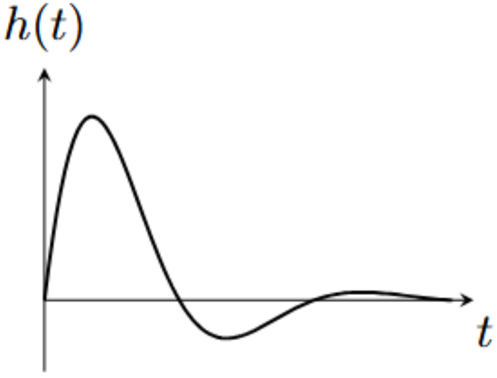


Laplace Transform

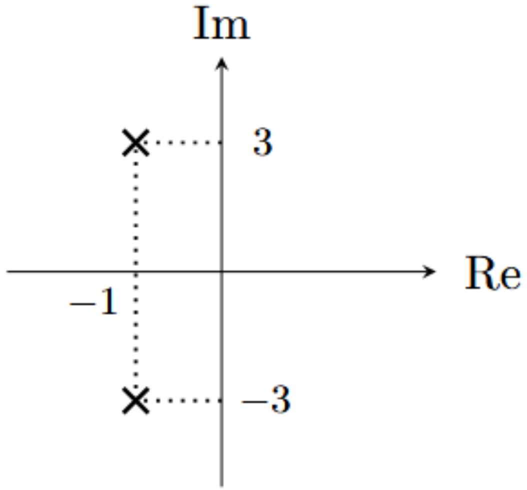
Another one

$$\boxed{h(t) = e^{-t}\sin(3t)}$$

$$\boxed{H(s) = \frac{3}{(s+3)^2+9}}$$



$$\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$$



Q&A Session / Done

Feedback

jschultev.github.io/personal_website/Feedback